

# Rational String Topology

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## Abstract

We use the computational power of rational homotopy theory to provide an explicit cochain model for the loop product and the string bracket of a 1-connected closed manifold  $M$ . We prove that the loop homology of  $M$  is isomorphic to the Hochschild cohomology of the commutative graded algebra  $A_{PL}(M)$  with coefficients in itself. Some explicit computations of the loop product and the string bracket are given.

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**Key words :** String homology, rational homotopy, Hochschild cohomology, free loop space, loop homology.

## 1 Introduction and preliminaries

Let  $M$  be a 1-connected closed oriented  $m$ -manifold and

$$LM = M^{S^1} := \{\text{unbased continuous map } S^1 \rightarrow M\}$$

be the associated free loop space. The *loop homology* of  $M$  is the ordinary homology of  $LM$ , with a shift of degrees,  $\mathbb{H}_*(LM) = H_{*+m}(LM)$  together with a commutative associative product, called *loop product*:

$$\mathbb{H}_p(LM) \otimes \mathbb{H}_q(LM) \rightarrow \mathbb{H}_{p+q}(LM), \quad a \otimes b \mapsto a \bullet b.$$

This product was discovered by M. Chas and D. Sullivan, [3] who defined it in terms of “transversal geometric chains” on  $LM$  by mixing the intersection product on  $M$ ,

$$H_p(M) \otimes H_q(M) \rightarrow H_{p+q-m}(M)$$

with the composition of loops

$$LM \times_M LM = \{\text{composable loop}\} \rightarrow LM.$$

In [5], R. Cohen and J.D.S. Jones used the Pontryagin-Thom structure to show that the loop product is realized at the homotopy level by Thom spectra. By using intersection product in the setting of Hilbert manifolds, [4], D. Chataur, provides an other description of the loop product.

In this paper we start with an elementary description of the loop product in terms of the diagonal class of  $M$ . Then, we prove

**Theorem A.** *When coefficients are taken in a field of characteristic zero, there exists an explicit model of the loop product in terms of the minimal model of Sullivan of  $M$ .*

This model is described in details in the text. Theorem A implies that the rational loop product does not depend on the geometry of the manifold but only on its rational homotopy type.

When coefficients are in a field, there is also an isomorphism of graded vector spaces, [5], [10] from the singular homology of the space  $LM$  to the Hochschild cohomology of the singular cochain complex of  $M$ :

$$\mathbb{H}_*(LM) \rightarrow HH^*(C^*M, C^*M).$$

In [5], Cohen and Jones assert that this isomorphism can be modified into an isomorphism of commutative graded algebras. More recently S. A. Merkulov, [15], gives a proof based on the theory of iterated integrals when the coefficients are the real numbers.

We construct here a direct isomorphism of algebras when the coefficients are the rational numbers. Our construction uses differential graded Lie algebras.

Recall that the Lie model of the space  $M$ , denoted  $L$ , is a differential graded Lie algebra defined by the property that the cochain algebra  $C^*L$  is quasi-isomorphic to the Sullivan model of  $M$ . In particular, if  $C_*(L)$  denotes the chain complex of the differential graded Lie algebra  $L$  then  $H_*(C_*(L)) = H_*(M; \mathbb{Q})$ .

The key point of our work is a chain morphism connecting the chain and the cochain complex of  $L$  with coefficients in  $(UL)_a^\vee = \text{Hom}(UL, \mathbb{Q})$  with the adjoint representation:

$$C^*(L; (UL)_a^\vee) \xrightarrow{\text{cap}} C_{m-*}(L; (UL)_a^\vee).$$

The morphism  $\text{cap}$  is the cap product with a cycle in  $C_m(L)$  representing the fundamental class in homology. Thanks to the diagonal on  $UL$ , the cochain complex  $C^*(L; (UL)_a^\vee)$  is a commutative differential graded algebra quasi-isomorphic to the Sullivan minimal model of  $LM$ . On the other hand, thanks to the multiplication in  $UL$ , the chain complex  $C_*(L; (UL)_a^\vee)$  is a differential graded coalgebra, and the cohomology of the dual differential graded algebra is isomorphic to the Hochschild cohomology  $HH^*(UL, UL)$ . More precisely we prove:

**Theorem B.** *Let  $L = \mathcal{L}_M$  be the minimal Lie model of  $M$ . Then the dual map  $\text{Hom}(\text{cap}, \mathbb{Q})$  induces in homology an isomorphism of algebras*

$$\mathbb{H}_*(LM) \xleftarrow{\cong} H^*(C^*(L; (UL)_a)),$$

where  $(UL)_a$  is equipped with the adjoint representation and  $H_*(LM)$  with the loop multiplication.

Now using homological algebra arguments, we prove that  $H^*(C^*(L; (UL)_a))$  is isomorphic to the Hochschild cohomology  $HH^*(UL, UL)$ . Then using the natural isomorphism

$$HH^*(UL, UL) \cong HH^*(C^*L, C^*L),$$

([9]), and the existence, for the minimal Lie model  $L$  of  $M$ , of quasi-isomorphisms of algebras,  $C^*L \xleftarrow{\cong} A \xrightarrow{\cong} C^*M$ , we recover

**Theorem C.** *Let  $M$  be a simply connected closed manifold, then we have a natural isomorphism of algebras  $\Phi : \mathbb{H}_*(LM) \xrightarrow{\cong} HH^*(C^*M, C^*M)$ .*

In [3], Chas and Sullivan construct also a morphism of algebras

$$I : \mathbb{H}_*(LM) \rightarrow H_*(\Omega M)$$

defined as follows. Let  $x_0$  be the base point of  $M$  and  $[x_0] \in H_0(LM)$  be the homology class of the constant loop, then for any  $a \in H_*(LM)$ , we define  $I(a) = a \bullet [x_0]$ . The map  $I$  looks like an intersection with the fiber  $\Omega M$ . We prove

**Theorem D.** *There exists an isomorphism of algebras  $\Psi$  making commutative the diagram*

$$\begin{array}{ccc} \mathbb{H}_*(LM) & \xrightarrow{\Phi} & HH^*(C^*M, C^*M) \\ I \downarrow & & \downarrow HH^*(C^*M, \varepsilon) \\ H_*(\Omega M) & \xrightarrow{\Psi} & HH^*(C^*M; \mathbb{Q}) \end{array}$$

where  $\varepsilon : C^*M \rightarrow \mathbb{Q}$  denotes the usual augmentation.

Let us recall that an element  $x \in \pi_q(M)$  is called a *Gottlieb element* ([8]-p.377), if the map  $x \vee id_M : S^q \vee M \rightarrow M$  extends to the product  $S^q \times M$ . The Gottlieb elements generate a subgroup  $G_*(M)$  of  $\pi_*(\Omega M)$  via the isomorphism  $\pi_*(\Omega M) \cong \pi_{*+1}(M)$ . Finally, we denote by  $\text{cat } M$  the Lusternik-Schnirelmann category of  $M$ , normalized so that  $\text{cat } S^n = 1$ . It follows from ([10]-Theorem 2) that :

- a) *The kernel of  $I$  is a nilpotent ideal of nilpotency index less than or equal to  $\text{cat } M$ .*
- b)  *$(\text{Im } I) \cap (\pi_*(\Omega M) \otimes \mathbb{Q}) = G_*(M) \otimes \mathbb{Q}$ .*
- c)  *$\sum_{i=0}^n \dim (\text{Im } I)_i \leq Cn^k$ , some constant  $C > 0$  and  $k \leq \text{cat } M$ .*

Our model of the loop product can also be used to compute the string bracket. Denote by  $LM \times_{S^1} ES^1$  the equivariant free loop space. The long exact homology sequence associated to the sphere bundle  $S^1 \rightarrow LM \times ES^1 \xrightarrow{p} LM \times_{S^1} ES^1$  has the form

$$\rightarrow H_n(LM) \xrightarrow{H_n(p)} H_n(LM \times_{S^1} ES^1) \xrightarrow{Cap} H_{n-2}(LM \times_{S^1} ES^1) \xrightarrow{M} H_{n-1}(LM) \rightarrow \dots$$

where  $Cap$  is the cap product with the characteristic class of the sphere bundle. Using this sequence, Chas and Sullivan give to  $H_*(LM \times_{S^1} ES^1)$  the structure of a graded Lie algebra of degree  $(2 - m)$

$$[a, b] = (-1)^{|a|} H_*(p)(M(a) \bullet M(b)), \quad ([3]).$$

Few things are known about this bracket. For surfaces of genus larger than zero, they recover formulas proved in the context of symplectic geometry.

Using computations made in [2] we prove that for any 1-connected manifold  $M$  such that the cohomology algebra  $H^*(M, \mathbb{Q})$  is monogenic, the string bracket is trivial. So for spheres, complex projective spaces..., string bracket is trivial. We also produce a model allowing explicit computations of the string bracket for any 1-connected compact manifold.

The text is divided in two parts. In the first one, after recalling some materials, we give a description of the loop product and the string bracket in terms of the Sullivan minimal model. This step makes the bridge between topology (geometry) and algebra. In the second part, we construct an explicit algebra isomorphism  $\mathbb{H}_*(LM) \cong HH^*(C^*M; C^*M)$ , and we prove Theorems B, C and D.

## Part 1 - Algebraic models

2. Preliminaries on differential homological algebra
3. A 'Sullivan model' description of the loop product
4. The string bracket

Part 2 - The isomorphism  $\mathbb{H}_*(LM) \cong HH^*(C^*M; C^*M)$

5. Hochschild cohomology of a Lie algebra
6. Lie models and the cap product  $C^*(L; M) \rightarrow C_{m-*}(L; M)$
7. Proof of Theorem B.
8. The intersection morphism

## 2 Preliminaries on differential homological algebra

All the graded vector spaces, algebras, coalgebras and Lie algebras  $V$  are defined over  $\mathbb{Q}$  and are supposed of finite type, i.e.  $\dim V_n < \infty$  for all  $n$ .

### 2.1 Graded vector spaces

If  $V = \{V_i\}_{i \in \mathbb{Z}}$  is a (lower) graded  $\mathbb{Q}$ -vector space (when we need upper graded vector space we put  $V_i = V^{-i}$  as usual) then  $V^\vee$  denotes the graded dual vector space,

$$V^\vee = \text{Hom}(V, \mathbb{Q}),$$

$sV$  denotes the suspension of  $V$ ,

$$(sV)_n = V_{n-1}, \quad (sV)^n = V^{n+1},$$

and  $TV$  denotes the tensor algebra on  $V$ , while we denote by  $TC(V)$  the free supplemented coalgebra generated by  $V$ .

Since we work with graded objects, we will make a special attention to signs. Recall that if  $P = \{P_i\}$  and  $N = \{N_i\}$  are differential graded vector spaces then

- $P \otimes N$  is a differential graded vector space :

$$(P \otimes N)_r = \bigoplus_{p+q=r} P_p \otimes N_q, \quad d_{P \otimes N} = d_P \otimes id_N + id_P \otimes d_N,$$

- $\text{Hom}(P, N)$  is a differential graded vector space :

$$\text{Hom}_n(P, N) = \prod_{k-l=n} \text{Hom}(P_l, N_k), \quad D_{\text{Hom}(P, N)} f = d_N \circ f - (-1)^{|f|} f \circ d_P.$$

If  $C$  is a differential graded coalgebra with diagonal  $\Delta$  and  $A$  is a differential graded algebra with product  $\mu$  then the cup product,  $f \cup g = \mu \circ (f \otimes g) \circ \Delta$ , gives to the differential graded vector space  $\text{Hom}(C, A)$  a structure of differential graded algebra.

### 2.2 Two-sided bar construction

Let  $(A, d)$  be a differential graded supplemented algebra,  $A = \mathbb{Q} \oplus \overline{A}$ , and  $(P, d)$  and  $(N, d)$  be differential graded  $A$ -modules, respectively right and left  $A$ -module. The *two-sided bar construction*,  $\mathcal{B}(P; A; N)$  is defined as follows:

$$\mathcal{B}_k(P; A; N) = P \otimes T^k(s\overline{A}) \otimes N.$$

A generic element is written  $p[a_1|a_2|\dots|a_k]n$  with degree  $|p| + |n| + \sum_{i=1}^k (|sa_i|)$ . The differential  $d = d_0 + d_1$  is defined by

$$d_0 : \mathcal{B}_k(P; A; N) \rightarrow \mathcal{B}_k(P; A; N), \quad d_1 : \mathcal{B}_k(P; A; N) \rightarrow \mathcal{B}_{k-1}(P; A; N),$$

with

$$d_0 p[\ ] n = d(p)[\ ] n + (-1)^{|p|} m[\ ] d(n), \quad d_1 p[\ ] n = 0$$

and if  $k > 0$ , and  $\epsilon_i = |p| + \sum_{j < i} (|s a_j|)$ :

$$\begin{aligned} d_0(p[a_1|a_2|\dots|a_k]n) &= d(p)[a_1|a_2|\dots|a_k]n - \sum_{i=1}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|d(a_i)|\dots|a_k]n \\ &\quad + (-1)^{\epsilon_{k+1}} p[a_1|a_2|\dots|a_k]d(n) \\ d_1 p[a_1|a_2|\dots|a_k]n &= (-1)^{|p|} p a_1[a_2|\dots|a_k]n + \sum_{i=2}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|a_{i-1}a_i|\dots|a_k]n \\ &\quad - (-1)^{\epsilon_k} p[a_1|a_2|\dots|a_{k-1}]a_k n \end{aligned}$$

The complex  $\mathcal{B}A = \mathcal{B}(\mathbb{Q}; A; \mathbb{Q})$  is a differential graded coalgebra whose comultiplication is defined by

$$\Delta[a_1|\dots|a_r] = \sum_{i=0}^r [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_r].$$

## 2.3 Hochschild cohomology

Let  $A$  be a differential graded algebra and  $N$  a differential graded  $A$ -bimodule. The *Hochschild cochain complex* of  $A$  with coefficients in  $M$  is the cochain complex

$$\mathbf{C}^*(A; N) = (\text{Hom}(TC(sA), N), D_0 + D_1) = \text{Hom}_{A^e}(\mathcal{B}(A; A; A), N),$$

with  $A^e = A \otimes A^{opp}$ . The differential  $D_0 + D_1$  comes from the natural differential on  $\text{Hom}_{A^e}(\mathcal{B}(A; A; A), N)$ :  $D_0(f)([\ ]) = d_N(f([\ ]))$ ,  $D_1(f)([\ ]) = 0$ , and for  $k \geq 1$  and  $f \in \text{Hom}(TC(sA), N)$ , then

$$D_0(f)([a_1|a_2|\dots|a_k]) = d_N(f([a_1|a_2|\dots|a_k])) + (-1)^{|f|} \sum_{i=1}^k (-1)^{\bar{\epsilon}_i} f([a_1|\dots|d_A a_i|\dots|a_k])$$

and

$$\begin{aligned} D_1(f)([a_1|a_2|\dots|a_k]) &= -(-1)^{|f|} \left( (-1)^{|a_1|+|f|} a_1 f([a_2|\dots|a_k]) \right) \\ &\quad (-1)^{|f|} \left( \sum_{i=2}^k (-1)^{\bar{\epsilon}_i} f([a_1|\dots|a_{i-1}a_i|\dots|a_k]) \right) \\ &\quad (-1)^{|f|} \left( +(-1)^{\bar{\epsilon}_k} f([a_1|a_2|\dots|a_{k-1}])a_k \right), \end{aligned}$$

where  $\bar{\epsilon}_i = |s a_1| + |s a_2| + \dots + |s a_{i-1}|$ .

The cohomology of  $\mathbf{C}^*(A; N)$  is called the *Hochschild cohomology of  $A$  with coefficients in  $N$*  and is denoted by  $HH^*(A; N)$ .

The natural cup product on  $\text{Hom}(TC(s\bar{A}), A)$  defined by

$$f \cup g = \mu_A \circ (f \otimes g) \circ \Delta_{TC(s\bar{A})},$$

makes  $\mathbf{C}^*(A; A)$  a differential graded algebra, and  $HH^*(A; A)$  into a commutative graded algebra ([11]).

## 2.4 The chain coalgebra and the cochain algebra on a differential graded Lie algebra

The *chain coalgebra* on a differential graded Lie algebra  $(L, d_L)$  is the graded differential coalgebra

$$C_*L = (\wedge sL, d_0 + d_1).$$

The differential and the comultiplication are defined by

$$d_0(sx_1 \wedge \cdots \wedge sx_k) = - \sum_{i=1}^k (-1)^{\sum_{j<i} |sx_j|} sx_1 \wedge \cdots \wedge sd_L x_i \wedge \cdots \wedge sx_k,$$

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = \sum_{1 \leq i < j \leq k} (-1)^{e_{ij}} s[x_i, x_j] \wedge \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k,$$

$$\Delta(sx_1 \wedge \cdots \wedge sx_k) = \sum_{j=0}^k \sum_{\sigma \in Sh(j)} \varepsilon_\sigma (sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(j)}) \otimes (sx_{\sigma(j+1)} \wedge \cdots \wedge sx_{\sigma(k)}),$$

where  $\varepsilon_\sigma$  is the usual sign associated to the graded permutation  $\sigma$ ,  $Sh(j)$  denotes the set of  $(j, k-j)$ -shuffles, and

$$e_{ij} = |sx_i|(1 + \sum_{k<i} |sx_k|) + |sx_j|(\sum_{k<j, k \neq i} |sx_k|).$$

For each left  $L$ -module  $N$  and right  $L$ -module  $P$  we can define the complex

$$C_*(P; L; N) = (P \otimes C_*L \otimes N, d_0 + d_1),$$

with

$$d_0(p \otimes c \otimes n) = dp \otimes c \otimes n + (-1)^{|p|} p \otimes d_0(c) \otimes n + (-1)^{|p|+|c|} p \otimes c \otimes dn,$$

$$\begin{aligned} d_1(p \otimes sx_1 \wedge \cdots \wedge sx_k \otimes n) = \\ (-1)^{|p|} p \otimes d_1(sx_1 \wedge \cdots \wedge sx_k) \otimes dn \\ + \sum_{i=1}^k (-1)^{|p|+(\sum_j |sx_j|)+|x_i|+|sx_i|(\sum_{j>i} |sx_j|)} p \otimes sx_1 \wedge \cdots \widehat{sx_i} \cdots \wedge sx_k \otimes x_i \cdot n \\ + \sum_{i=1}^k (-1)^{|p|+|sx_i|(\sum_{j<i} |sx_j|)} p \cdot x_i \otimes sx_1 \wedge \cdots \widehat{sx_i} \cdots \wedge sx_k \otimes n. \end{aligned}$$

The chain complex of  $L$  with coefficients in a left-module  $N$ ,

$$C_*(L; N) := C_*(\mathbb{Q}; L; N)$$

is a left  $C_*L$ -comodule. In a similar way the chain complex  $C_*(P; L) := C_*(P; L; \mathbb{Q})$  is a  $C_*L$ -right comodule. Here  $\mathbb{Q}$  is equipped with the trivial action.

When  $N = UL$  with the action induced by left multiplication then  $C_*(L, UL)$  is a left  $C_*L$ -comodule and a right  $UL$ -module and both structures are compatible. Moreover the augmentation

$$C_*(L; UL) = C_*L \otimes UL \xrightarrow{\varepsilon \otimes \varepsilon} \mathbb{Q}$$

is a quasi-isomorphism.

The *cochain complex* of  $L$  with coefficients in a right  $L$ -module  $R$  is defined by

$$C^*(L; R) = \text{Hom}_{UL}(C_*(L; UL), R).$$

The homology and the cohomology of  $L$  with coefficients in a module  $N$  are defined by  $H_*(L; N) = H_*(C_*(L; N))$  and  $H^*(L; N) = H^*(C^*(L; N))$ .

The *cochain algebra* on a differential graded Lie algebra  $(L, d_L)$  is the graded differential algebra  $C^*L = C^*(L; \mathbb{Q}) = \text{Hom}(C_*L, \mathbb{Q})$ .

## 2.5 Semifree resolutions

Let  $A$  be a differential graded algebra. A module  $P$  is called a *semifree module* if  $P$  is equipped with a filtration  $P = \cup_{n \geq 0} P(n)$ , satisfying  $P(0) = 0$ ,  $P(n) \subset P(n+1)$  and such that  $P(n)/P(n-1)$  is free on a basis of cycles ([8]).

For any  $A$ -module  $N$ , there exists a semifree module  $P$  and a quasi-isomorphism  $\varphi : P \rightarrow N$ . The module  $P$  is called a semifree resolution of  $N$ .

For instance the complex  $\mathcal{B}(A; A; A) \xrightarrow{\sim} A$  is a semifree resolution of  $A$  as an  $A^e$ -module [8, 4.3(ii)].

The complex  $C_*(UL; L; UL) = UL \otimes C_* L \otimes UL$  is a semifree resolution of  $UL$  as an  $UL$ -bimodule. This follows from the quasi-isomorphism of  $UL$ -bimodules  $\theta : C_*(UL; L; UL) \rightarrow UL$  defined by  $\theta(a \otimes 1 \otimes b) = ab$  and  $\theta(a \otimes c \otimes b) = 0$  for  $a, b \in UL$  and  $c \in \wedge^+(sL)$ .

### 3 A 'Sullivan model' description of the loop product

### 3.1 A definition of the loop product at the homological level

Let  $M$  be a closed riemannian  $m$ -dimensional manifold and  $\Delta M \hookrightarrow M \times M$  be the diagonal embedding of  $M$  in  $M \times M$ . We denote by  $\nu$  the normal bundle of  $\Delta M$  in  $M \times M$  which is identified with the tangent bundle  $\tau M$  of  $M$ . The exponential map of the normal bundle  $\nu$ , defined only on a neighborhood of the zero section of  $\nu$ , induces a diffeomorphism from a  $m$ -dimensional disk bundle,  $\tau_D M$ , onto a tubular neighborhood, denoted  $N$ , of  $\Delta M$ . The associated  $(m-1)$ -sphere bundle,  $\tau_S M$  is diffeomorphic to the boundary  $\partial N$  of  $N$ .

Let  $p : L(M \times M) = LM \times LM \rightarrow M \times M$   $p(\omega, \omega') = (\omega(0), \omega'(0))$  be the product of the two loop fibrations. We consider the commutative diagram

$$\begin{array}{ccccccc}
LM \times_M LM = p^{-1}(\Delta M) & \xrightarrow{j'} & p^{-1}(N) & \xrightarrow{i'} & LM \times LM & \xleftarrow{k'} & p^{-1}(M \times M - \Delta M) \\
\downarrow q & & \downarrow p' & & \downarrow p & & \downarrow p'' \\
M = \Delta M & \xrightarrow{j} & N & \xrightarrow{i} & M \times M & \xleftarrow{k} & M \times M - \Delta M,
\end{array}$$

where  $i, j, k, i', j'$  and  $k'$  are the natural inclusions.

One observes that in the above diagram the vertical maps  $p', q$  and  $p''$  are pullback fibrations and that  $p^{-1}(N)$  can be identified with the pullback of the fiber bundle  $\tau_D M$  along  $q$ :  $p^{-1}(N) \cong q^* \tau_D M$ .

The loop multiplication on  $H_*(LM)$  with coefficients in  $\mathbb{Q}$  is then defined by the composition of maps

$$\begin{array}{ccc}
H_*(LM) \otimes H_*(LM) & & \\
\cong \downarrow & & \\
H_*(LM \times LM) \longrightarrow & H_*(LM \times LM, p^{-1}(M \times M - \Delta M)) & \\
& E \downarrow \cong & \\
& H_*(p^{-1}(N), p^{-1}(\partial N)) & \\
& \cong \downarrow & \\
& H_*(q^* \tau_D M, q^* \tau_S M) & \\
& Th_* \downarrow & \\
H_{*-m}(LM \times_M LM) & \xrightarrow{H_{*-m}(\gamma)} & H_{*-m}(LM)
\end{array}$$

Here  $\gamma : LM \times_M LM \rightarrow LM$  denotes the composition of loops,  $E$  is the classical excision isomorphism and  $Th_*$  is the Thom isomorphism recalled below.

We recall that, if  $p : X \rightarrow M$  is a  $m$ -dimensional vector bundle with associated disk bundle,  $X_D$ , and sphere bundle  $X_S$ , there exists a class  $\alpha \in H^m(X_D, X_S)$ , called the Thom class of the disk bundle, such that the cap product with  $\alpha$  followed by the projection on  $M$  induces an isomorphism  $Th_* : H_*(X_D, X_S) \rightarrow H_{*-m}(M)$ . Here we consider the normal bundle  $\nu$  defined above and its  $m$ -dimensional disk bundle  $\tau_D M$ , that defines the Thom class  $\alpha \in H^m(\tau_D M, \tau_S M)$ . The isomorphism  $Th_* : H_*(q^* \tau_D M, q^* \tau_S M) \rightarrow H_{*-m}(LM \times_M LM)$  is the cap product with  $q^* \alpha$  followed by the projection.

### 3.2 The Sullivan minimal model of $M$

The Sullivan minimal model of a simply connected closed  $m$ -dimensional manifold  $M$ ,  $\mathcal{M}_M$ , is a commutative differential graded algebra defined over the rational numbers that represents the rational homotopy type of  $M$  ([18],[8]). As an algebra  $\mathcal{M}_M$  is a free graded commutative algebra,

$$\mathcal{M}_M = \wedge V,$$

with  $V^n \cong \text{Hom}(\pi_n(M), \mathbb{Q})$ . The differential  $d$  satisfies the minimality condition  $d(V) \subset \wedge^{\geq 2}(V)$ .

The minimal model is unique up to isomorphism. It depends only on the rational homotopy type of  $M$  and contains all the rational homotopy informations of  $M$ . In particular there exists a differential graded algebra  $B$  and quasi-isomorphisms

$$\mathcal{M}_M \xleftarrow{\sim} B \xrightarrow{\sim} C^*(M; \mathbb{Q}).$$

There exists also a quasi-isomorphism of differential graded algebras  $\psi : \mathcal{M}_M \otimes \mathbb{R} \rightarrow \Omega_{DR} M$ , where  $\Omega_{DR} M$  denotes the algebra of de Rham forms on  $M$ .

More generally a Sullivan (or free) model for  $M$  is a commutative differential graded algebra  $(\wedge W, d)$ , that is free as a commutative graded algebra, for which there exists a quasi-isomorphism  $\varphi : \mathcal{M}_M \rightarrow (\wedge W, d)$ , with  $W = W^{\geq 2}$  a finite type graded vector space. It is sometimes more convenient to work with free models having special properties than to work with minimal models, as we will see further.

### 3.3 Examples of models

Let  $(\wedge V, d)$  be a free model of  $M$ . A free model for the free loop space  $LM$  is given by  $(\wedge V \otimes \wedge sV, D)$ , where  $(sV)^n = V^{n+1}$ ,  $D(sv) = -s(dv)$ , where  $s$  has been extended to  $\wedge V \otimes \wedge sV$  as a derivation satisfying  $s(v) = sv$  and  $s(sv) = 0$  ([20]).

A model of  $LM \times_M LM$  is then given by  $(\wedge V \otimes \wedge sV \otimes \wedge s'V, D)$  where  $(s'V)^n = V^{n+1}$  and  $D(s'v) = -s'(dv)$ . In other words,  $(\wedge V \otimes \wedge s'V, D)$  is a copy of  $(\wedge V \otimes \wedge sV, D)$  and

$$(\wedge V \otimes \wedge sV \otimes \wedge s'V, D) = (\wedge V \otimes \wedge sV, D) \otimes_{\wedge V} (\wedge V \otimes \wedge s'V, D).$$

A model for the diagonal map  $\Delta : M \rightarrow M \times M$  is given by the multiplication  $\mu : (\wedge V, d) \otimes (\wedge V', d) \rightarrow (\wedge V, d)$  where  $(\wedge V', d) = (\wedge V, d)$ . The Sullivan relative model of  $\mu$ , is a commutative diagram in which  $\varphi$  is a quasi-isomorphism and  $i$  the canonical inclusion

$$\begin{array}{ccc} (\wedge V, d) \otimes (\wedge V', d) & \xrightarrow{\mu} & (\wedge V, d) \\ & \searrow i & \uparrow \varphi \\ & & (\wedge V \otimes \wedge V' \otimes \wedge \bar{V}, D). \end{array}$$

Here  $\bar{V}^n = V^{n+1}$  and  $D(\bar{v}) = (v - v') \in \bar{V} \oplus \wedge^{\geq 2}(V \oplus V' \oplus \bar{V})$ .



### 3.4 A Sullivan model for the loop product

In the next section, we will construct an explicit model for the composition of paths  $LM \times_M LM \rightarrow LM$  that has the form

$$c : (\wedge V \otimes \wedge sV, D) \rightarrow (\wedge V \otimes \wedge sV \otimes \wedge s'V, D).$$

Denote now by  $u_M$  a cycle in  $\text{Hom}(\wedge V, \mathbb{Q})$  representing the fundamental class in homology:  $[u_M] \in H_m(M; \mathbb{Q}) = H_m(\text{Hom}(\wedge V, \mathbb{Q}))$ . We choose an homogeneous basis  $\alpha_i$  of  $H^*(M)$  and its Poincaré dual basis  $\alpha_i^\#$ :

$$\langle \alpha_i \cup \alpha_j^\#; [u_M] \rangle = \delta_{ij}.$$

We choose then cocycles  $a_i$  and  $a_i^\#$  in  $\wedge V$  whose cohomology classes are respectively  $\alpha_i$  and  $\alpha_i^\#$ . This gives the cocycle

$$T = \sum_i (-1)^{|a_i|} a_i \otimes a_i^\# \in \wedge V \otimes \wedge V.$$

By Theorem 1 below, the multiplication by  $T$ ,

$$\mu_T : (\wedge V, d) \otimes (\wedge V', d) \rightarrow (\wedge V, d) \otimes (\wedge V', d)$$

extends into a morphism of  $(\wedge V \otimes \wedge V')$  differential modules

$$\mu_T : (\wedge V \otimes \wedge V' \otimes \wedge \bar{V}, D) \rightarrow (\wedge V, d) \otimes (\wedge V', d).$$

We have thus defined all the terms involved in the diagram

$$\begin{array}{ccc} (\wedge V \otimes \wedge sV, D) & & \\ \downarrow c & & \\ (\wedge V \otimes (\wedge sV \otimes \wedge s'V), D) & \xleftarrow{\varphi^{\otimes 1}} & (\wedge V \otimes \wedge V' \otimes \wedge \bar{V} \otimes (\wedge sV \otimes \wedge s'V), D') \\ & & \downarrow \mu_T \otimes 1 \\ & & (\wedge V \otimes \wedge sV, D) \otimes (\wedge V' \otimes \wedge s'V, D) \end{array}$$

Theorem A of the Introduction can be now expressed in more explicit terms

**Theorem A.** *The induced map in cohomology*

$$H^*(\mu_T \otimes 1) \circ H^*(\varphi \otimes 1)^{-1} \circ H^*(c) : H^{*-m}(LM) \rightarrow H^*(LM) \otimes H^*(LM)$$

is the coproduct on  $H^*(LM)$  dual to the loop product on  $H_*(LM)$ .

**Proof.** We have to compute the map induced in cohomology by the composite (1)

$$\begin{array}{ccc} C^{*-m}(LM \times_M LM) & & \\ \cup \beta \downarrow & & \\ C^*(q^* \tau_D M, q^* \tau_S M) & \xrightarrow{\cong} & C^*(p^{-1}(N), p^{-1}(\partial N)) \xrightarrow{E} C^*(LM \times LM, p^{-1}(M \times M - \Delta M)) \\ & & \downarrow \\ & & C^*(LM \times LM) \end{array}$$

where  $\beta$  is the Thom class of the disk bundle  $p^{-1}(N) \rightarrow LM \times_M LM$ . Since this fiber bundle is the inverse image of the disk bundle  $N \rightarrow M$ ,  $\beta = q^*(\beta')$ , where  $\beta'$  is the Thom

class of  $\tau_D M$ . We observe that the diagram (1) is obtained by pullbacking  $p$  along the maps arising in the diagram

$$(2) \quad \begin{array}{c} C^{*-m}(M) \\ \cup \beta' \downarrow \\ C^*(\tau_D M, \tau_S M) \end{array} \xrightarrow{\cong} C^*(N, \partial N) \xrightarrow{E} C^*(M \times M, (M \times M - \Delta M)) \rightarrow C^*(M \times M).$$

A model for the composite (1), as a  $(\wedge V \otimes \wedge sV) \otimes (\wedge V' \otimes \wedge s'V)$ -module is thus obtained by making the tensor product of a model of (2) with  $(\wedge V \otimes \wedge sV) \otimes (\wedge V' \otimes \wedge s'V)$  over  $\wedge V \otimes \wedge V'$ . Now it follows from ([1], 11.2) that the map induced in cohomology by the composite (2) is the cohomology intersection, i.e. the multiplication by  $[T]$ . Therefore by Theorem 1 below, a model of (1) is given by  $\mu_T$ . This ends the proof of Theorem A.  $\square$

**Theorem 1.** *There exists up to homotopy one and only one morphism of differential graded  $(\wedge V \otimes \wedge V')$ -module*

$$f : (\wedge V \otimes \wedge V' \otimes \wedge \bar{V}, D) \rightarrow (\wedge V, d) \otimes (\wedge V', d)$$

whose restriction to  $\wedge V \otimes \wedge V'$  gives in homology the multiplication by  $T$ .

**Proof.** The commutation with the differentials gives to the graded vector space

$$\text{Hom}_{\wedge V \otimes \wedge V'}(\wedge V \otimes \wedge V' \otimes \wedge \bar{V}, \wedge V \otimes \wedge V')$$

a differential whose homology classes in degree  $d$  are the homotopy classes of module morphisms of degree  $d$ . By definition of  $\text{Ext}$ , we have,

$$H^*(\text{Hom}_{\wedge V \otimes \wedge V'}(\wedge V \otimes \wedge V' \otimes \wedge \bar{V}, \wedge V \otimes \wedge V')) = \text{Ext}_{\wedge V \otimes \wedge V'}^*(\wedge V, \wedge V \otimes \wedge V')$$

where  $\wedge V$  is viewed as a  $\wedge V \otimes \wedge V'$ -module by the multiplication map.

We use the Moore spectral sequence and consider only the total degree

$$\text{Ext}_{H \otimes H}^*(H, H \otimes H) \Rightarrow \text{Ext}_{\wedge V \otimes \wedge V'}^*(\wedge V, \wedge V \otimes \wedge V'),$$

with  $H = H(\wedge V, d)$ .

Since  $H \otimes H$  is a Poincaré duality algebra,  $\text{Ext}_{H \otimes H}^q(\mathbb{Q}, H \otimes H) = 0$  for  $q \neq 2m$  and  $\dim \text{Ext}_{H \otimes H}^{2m}(\mathbb{Q}, H \otimes H) = 1$ . Therefore by induction on the dimension, we have that

$$\text{Ext}_{H \otimes H}(E, H \otimes H) = \text{Ext}_{H \otimes H}^{\geq 2m-d}(E, H \otimes H)$$

when  $E$  is finite dimensional and  $E = E^{\leq d}$ . Since  $M$  is simply connected,  $H^{m-1} = 0$ , and the long exact sequence associated to the short exact sequence  $0 \rightarrow H^m \rightarrow H \rightarrow H/H^m \rightarrow 0$  gives  $\text{Ext}_{H \otimes H}^m(H, H \otimes H) \cong \mathbb{Q}\alpha$ , for some  $\alpha$ , and  $\text{Ext}_{H \otimes H}^p(H, H \otimes H) = 0$  for  $p = m-1$  and  $p = m+1$ . Therefore the element  $\alpha$  will be an  $\infty$ -cycle and never a boundary. In particular

$$\text{Ext}_{\wedge V \otimes \wedge V'}^m(\wedge V, \wedge V \otimes \wedge V') \cong \mathbb{Q}.$$

Since the multiplication by  $T$ ,  $H \rightarrow H \otimes H$  is a morphism of  $H \otimes H$ -bimodules, this multiplication is the representant of a generator of  $\text{Ext}_{H \otimes H}^m(H, H \otimes H)$ .

This proves the existence and the unicity of a map  $f$  that extends the multiplication by  $T$  on  $\wedge V \otimes \wedge V'$ .  $\square$

Remark that the existence of  $\mu_T$  and the construction of the model of the composite can also be deduced from a more general construction given by Lambrechts and Stanley in [13].

### 3.5 Construction of a model for $LM \times_M LM \rightarrow LM$

The free loop space  $LM$  is the pullback of the diagram

$$\begin{array}{ccc} LM & \rightarrow & M^{[0,1]} \\ \downarrow & & \downarrow (p_0, p_1) \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where  $p_i(\omega) = \omega(i)$ . In the same way, the space  $LM \times_M LM$  is the pullback of the diagram

$$\begin{array}{ccc} LM \times_M LM & \rightarrow & M^{[0,1]} \times_M M^{[0,1]} \\ \downarrow & & \downarrow q \\ M & \xrightarrow{\Delta} & M \times M \times M, \end{array}$$

where  $q(\omega, \omega') = (\omega(0), \omega(1) = \omega'(0), \omega'(1))$ . The composition of paths  $LM \times_M LM \rightarrow LM$  is the pullback of  $id_M$  and the path composition  $\nu$  over  $\rho$ :

$$\begin{array}{ccccc} LM \times_M LM & \xrightarrow{\quad} & M^{[0,1]} \times_M M^{[0,1]} & & \\ \downarrow & \searrow & \downarrow \nu & & \\ & LM & \xrightarrow{\quad} & M^{[0,1]} & \\ \downarrow & \downarrow & \downarrow q & & \downarrow (p_0, p_1) \\ M & \xrightarrow{\Delta} & M \times M \times M & \xrightarrow{\rho} & M \times M \\ \downarrow id_M & \downarrow & \downarrow \Delta & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M & & \end{array}$$

Here  $\rho(\alpha, \beta, \gamma) = (\alpha, \gamma)$ .

Denote now by  $\sigma : M \rightarrow M^{[0,1]}$  the map sending a point to the constant path at that point. Then  $\sigma$  is an homotopy equivalence making commutative the diagram

$$\begin{array}{ccccc} M^{[0,1]} \times_M M^{[0,1]} & \xleftarrow{(\sigma, \sigma)} & M & \xrightarrow{id_M} & M \\ \downarrow q & \swarrow \nu & \downarrow \sigma & \swarrow \sigma & \\ & M^{[0,1]} & \xleftarrow{\Delta} & M & \\ \downarrow \Delta & \downarrow (p_0, p_1) & \downarrow \Delta & \downarrow \Delta & \\ M \times M \times M & \xrightarrow{\rho} & M \times M & \xrightarrow{\rho} & M \times M \end{array}$$

Therefore a model of  $\nu$  is a model of  $id_M$ . A model of the loop composition  $LM \times_M LM \rightarrow LM$  is then obtained by the tensor product over a model of  $\rho$  of a model of  $id_M$  and a relative model of  $id_M$ . Denote by

$$\begin{array}{ccc} (\wedge V \otimes \wedge V' \otimes \wedge sV, D) & \xrightarrow{\varphi} & (\wedge V, d) \\ \uparrow i & \nearrow \mu & \\ (\wedge V \otimes \wedge V', D) & & \end{array}$$

a model of the multiplication  $\mu$  as defined in 3.3:  $(\wedge V', d)$  is a copy of  $(\wedge V, d)$ ,  $(sV)^n = V^{n+1}$ ,  $D(sv) = v + v' \in \wedge^{\geq 2}(V \oplus V' \oplus sV) \oplus sV$ ,  $\mu(v) = \mu(v') = v$  and  $\varphi(sv) = 0$ .

We first construct a relative model  $c'$  of  $\nu$ .

$$\begin{array}{ccccc}
& & \xrightarrow{\varphi \otimes \wedge V \varphi} & \wedge V & \\
& \nearrow c' & \mu & \nwarrow id_{\wedge V} & \\
Z & \xrightarrow{i} & \wedge V \otimes \wedge V' \otimes \wedge sV & \xrightarrow{\varphi} & \wedge V \\
& \nwarrow \gamma & \uparrow i & \mu & \nearrow \\
& \wedge V \otimes \wedge V' \otimes \wedge V'' & \wedge V \otimes \wedge V' & & 
\end{array}$$

Here  $Z = (\wedge(V \oplus V' \oplus V'' \oplus sV \oplus s'V), D)$ ,  $\gamma(a \otimes b') = a \otimes b''$ ,  $(\wedge V' \otimes \wedge V'' \otimes \wedge s'V, D)$  is a copy of  $(\wedge V \otimes \wedge V' \otimes \wedge sV, D)$  and

$$(\wedge(V \oplus V' \oplus V'' \oplus sV \oplus s'V), D) = (\wedge(V \oplus V' \oplus sV), D) \otimes_{\wedge V'} (\wedge(V' \oplus V'' \oplus s'V), D).$$

A model  $c$  for the path composition is then obtained by the tensor product of  $c'$  with  $(\wedge V, d)$  over  $\wedge V \otimes \wedge V' \otimes \wedge V''$ .

$$\begin{array}{ccc}
(\wedge V \otimes \wedge V' \otimes \wedge sV, D) & \xrightarrow{c'} & (\wedge(V \oplus V' \oplus V'' \oplus sV \oplus s'V), D) \\
\pi \downarrow & & \pi \downarrow \\
(\wedge V \otimes \wedge sV, D) & \xrightarrow{c} & (\wedge V \otimes \wedge sV \otimes \wedge s'V, D)
\end{array}$$

Here  $\pi(v) = \pi(v') = \pi(v'') = v$ ,  $\pi(sv) = sv$  and  $\pi(s'v) = s'v$ .

### 3.6 The example of formal spaces

Let  $M$  be a formal space, i.e. a space  $M$  whose minimal model  $(\wedge V, d)$  is quasi-isomorphic to  $(H^*(M), 0)$ . Examples of formal spaces are given by simply connected compact Kähler manifolds ([7]) and homogeneous spaces of rank zero (quotient of compact connected Lie groups by closed subgroups of the same rank).

Denote by  $(\wedge V \otimes \wedge \bar{V}, D)$  the minimal model of the free loop space  $LM$ . When  $M$  is a formal space the coproduct dual to the loop product is the map in cohomology induced by the composition

$$(H^*(M) \otimes \wedge \bar{V}, D) \xrightarrow{c} (H^*(M) \otimes \wedge(\bar{V} \oplus \bar{V}'), D) \xrightarrow{\mu_T \otimes 1} (H^*(M) \otimes \wedge \bar{V}, D) \otimes (H^*(M) \otimes \wedge \bar{V}, D).$$

with  $(H^*(M) \otimes \wedge \bar{V}, D) = H^*(M) \otimes_{\wedge V} (\wedge V \otimes \wedge \bar{V}, D)$ , and where  $c$  is a model for the path composition  $LM \times_M LM \rightarrow LM$ .

**The particular case  $M = \mathbb{C}P^n$ .** This is a formal space whose minimal model is given by  $(\wedge(x, y), d)$ ,  $d(y) = x^{n+1}$ ,  $|x| = 2$ ,  $|y| = 2n + 1$ .

The model of the free loop space is

$$(\wedge(x, \bar{x}, y, \bar{y}), D), \quad D(\bar{x}) = 0, D(\bar{y}) = -(n+1)x^n \bar{x}.$$

Since we have a quasi-isomorphism  $(\wedge(x, \bar{x}, y, \bar{y}), D) \rightarrow (\wedge(x, \bar{x}, \bar{y})/(x^{n+1}), d)$ , the cohomology of the free loop space is

$$H^*(\wedge(x, \bar{x}, \bar{y})/(y^{n+1}), D) \cong \mathbb{Q} \cdot 1 \oplus \left( \wedge^+(x, \bar{x})/(x^{n+1}, x^n \bar{x}) \otimes \wedge \bar{y} \right).$$

A model  $c$  of the path composition  $LM \times_M LM \rightarrow LM$  is given by

$$c(\bar{x}) = \bar{x} + \bar{x}', \quad c(\bar{y}) = \bar{y} + \bar{y}' - \frac{n(n+1)}{2} x^{n-1} \bar{x} \bar{x}'.$$

The dual of the loop product is quite easy to handle and is induced by the map

$$\theta : (H^*(M) \otimes \wedge(\bar{x}, \bar{y}), D) \rightarrow (H^*(M) \otimes \wedge(\bar{x}, \bar{y}), D) \otimes (H^*(M) \otimes \wedge(\bar{x}, \bar{y}), D),$$

$$\left\{ \begin{array}{l} \theta(\alpha \otimes \bar{y}^{[s]}) = \sum_{p=0}^n \sum_{j=0}^s \alpha x^p \bar{y}^{[j]} \otimes x^{n-p} \bar{y}^{[s-j]} \\ \quad - \frac{n(n+1)}{2} \sum_{p=0}^n \sum_{j=0}^{s-1} \alpha x^{n-1+p} \bar{x} \bar{y}^{[j]} \otimes x^{n-p} \bar{x} \bar{y}^{[s-j]}, \\ \theta(\alpha \otimes \bar{x} \otimes \bar{y}^{[s]}) = (1 \otimes \bar{x} + \bar{x} \otimes 1) \cdot \left( \sum_{p=0}^n \sum_{j=0}^s \alpha x^p \bar{y}^{[j]} \otimes x^{n-p} \bar{y}^{[s-j]} \right), \end{array} \right.$$

with  $\alpha \in H^*(M)$  and  $\bar{y}^{[s]} = \bar{y}^s / s!$ .

We consider the basis for the free loop space homology formed by the elements

$$1, a_{p,q}, b_{r,s}, \quad p = 1, \dots, n, \quad q \geq 0, \quad s \geq 0, \quad r = 0, \dots, n-1,$$

with  $|a_{p,q}| = 2p + 2qn$ ,  $|b_{r,s}| = 2r + 1 + 2sn$ , corresponding by the isomorphism  $H^*(LM) = \text{Hom}(H_*(LM), \mathbb{Q})$  to the classes 1,  $x^p \bar{y}^{[q]}$ , and  $x^p \bar{x} \bar{y}^{[q]}$ .

$$\langle a_{p,q}, x^r \bar{y}^{[s]} \rangle = \begin{cases} 1 & \text{if } (p,q) = (r,s) \\ 0 & \text{otherwise} \end{cases}, \quad \langle b_{p,q}, x^r \bar{x} \bar{y}^{[s]} \rangle = \begin{cases} 1 & \text{if } (p,q) = (r,s) \\ 0 & \text{otherwise} \end{cases}$$

From the description of the map  $\theta$ , we deduce that the loop product is described by the formulas

$$a_{p,q} \bullet a_{r,s} = a_{p+r-n, q+s}, \quad a_{p,q} \bullet b_{r,s} = b_{p+r-n, q+s}, \quad (a_{n-1,0})^n = 1, \quad 1 \bullet a_{n,1} = 0.$$

This shows that

$$\mathbb{H}_*(L(\mathbb{C}P^n); \mathbb{Q}) \cong \wedge(a, b, t) / (a^{n+1}, a^n b, a^n t),$$

with  $|a| = -2$ ,  $|b| = -1$  and  $|t| = 2n$ ,  $a = a_{n-1,0}$ ,  $b = b_{n-1,0}$ ,  $t = a_{n,1}$ . Note that this computation can also be found in [6] as an application of a spectral sequence defined by Cohen, Jones and Yan.

## 4 The string bracket

Following Chas and Sullivan [3], we call string homology the equivariant homology of the free loop space

$$\mathcal{H}_* = H_{*+m}^{S^1}(LM) = H_{*+m}(LM \times_{S^1} ES^1; \mathbb{Q}).$$

The circle fibration  $S^1 \rightarrow LM \times ES^1 \xrightarrow{p} LM \times_{S^1} ES^1$  leads to the exact Gysin sequence

$$\cdots \rightarrow \mathbb{H}_n \xrightarrow{H(p)} \mathcal{H}_n \xrightarrow{c} \mathcal{H}_{n-2} \xrightarrow{\mathbb{M}} \mathbb{H}_{n-1} \rightarrow \cdots$$

where  $c$  is the cap product with the characteristic class of the circle bundle.

**Definition.** The string bracket on  $\mathcal{H}_*$  is the binary operation  $[-, -] : \mathcal{H}_* \otimes \mathcal{H}_* \rightarrow \mathcal{H}_{*-2}$  defined by

$$[a, b] = (-1)^{|a|} H_*(p)(\mathbb{M}(a) \bullet \mathbb{M}(b)),$$

where the product  $\bullet$  is the loop product on  $\mathbb{H}_*(LM)$ . Chas and Sullivan prove that  $(\mathcal{H}_*, [-, -])$  is a graded Lie algebra of degree 2 ([3], Theorem 6.1).

Using results on the rational cohomology of  $LM \times_{S^1} ES^1$ , [2], and the splitting of the long exact Gysin sequence in cohomology for formal spaces, [19], it is easy to prove:

**Theorem 2.** *Let  $M$  be a 1-connected closed manifold such that  $H^*(M, \mathbb{Q})$  is a truncated algebra in one generator, then the string bracket on the string homology  $\mathcal{H}_*$  is trivial.*

**Proof.**

From [2], we have the following facts:

If  $H^*(M) = \Lambda u$  with  $|u| = 2p + 1$  then  $\mathcal{H}_{2i} = 0$  for all  $i$ .

If  $H^*(M) = \Lambda u/(u^{n+1})$  with  $|u| = 2p$ , then  $\dim_{\mathbb{Q}} \mathcal{H}_{2i} = 1$  for all  $i$ . Furthermore, the space  $M$  is formal and it is shown in [19] that the map  $c$  is an isomorphism.

In the two cases above, we get the nullity of the maps:

$$\mathbb{E} = H_*(p) : \mathbb{H}_{2i} \longrightarrow \mathcal{H}_{2i}$$

$$\mathbb{M} : \mathcal{H}_{2i} \longrightarrow \mathbb{H}_{2i+1}$$

Let  $a \in \mathcal{H}_{2i-1}$  and  $b \in \mathcal{H}_{2b-1}$ , for some  $(i, j) \in \mathbb{Z}^2$ , then  $\mathbb{M}(a) \bullet \mathbb{M}(b) \in \mathbb{H}_{2(i+j+1)}$ , so we have

$$[a, b] = -\mathbb{E}(\mathbb{M}(a) \bullet \mathbb{M}(b)) = 0.$$

□

**Description of the Gysin sequence.** When  $(\wedge V, d)$  denotes the minimal model of  $M$ , the Gysin sequence in cohomology can be easily computed using models ([20], [2]). First of all a model of  $LM$  is  $(\wedge V \otimes \wedge(sV), D)$  with  $D(sv) = -s(dv)$ . Furthermore a model for the equivariant free loop space  $LM \times_{S^1} ES^1$  is given by the commutative differential graded algebra

$$(\wedge V \otimes \wedge sV \otimes \wedge u, D), \quad D(u) = 0, |u| = 2, D(v) = d(v) + us(v), D(sv) = -s(dv).$$

A model for the inclusion  $LM \rightarrow LM \times_{S^1} ES^1$  is given by the projection

$$\pi : (\wedge V \otimes \wedge sV \otimes \wedge u, D) \rightarrow (\wedge V \otimes \wedge sV, D)$$

defined by  $\pi(u) = 0$ . The map  $s$  is the map induced in cohomology by the derivation defined by  $s(v) = sv$  and  $s(sv) = 0$  for all  $v \in V$ . The Gysin sequence in cohomology is then identified with a long exact sequence defined in terms of the Sullivan models

$$\begin{array}{ccc} H^n(LM) & \xrightarrow{\cong} & H^n(\wedge(V \oplus sV), D) \\ \uparrow H^n(p) & & \uparrow H^n(\pi) \\ H^n(LM \times_{S^1} ES^1) & \xrightarrow{\cong} & H^n(\wedge(V \oplus sV \oplus \mathbb{Q}u), D) \\ \uparrow c' & & \uparrow \cup u \\ H^{n-2}(LM \times_{S^1} ES^1) & \xrightarrow{\cong} & H^{n-2}(\wedge(V \oplus sV \oplus \mathbb{Q}u), D) \\ \uparrow M' & & \uparrow s \\ H^{n-1}(LM) & \xrightarrow{\cong} & H^{n-1}(\wedge(V \oplus sV), D) \end{array}$$

Therefore we have proved

**Theorem 3.** *With rational coefficients, the string bracket is dual to the linear map  $b^\vee : H_{S^1}^*(LM) \rightarrow H_{S^1}^*(LM) \otimes H_{S^1}^*(LM)$  defined by the composition*

$$H_{S^1}^*(LM) \xrightarrow{H^*(\pi)} H^*(LM) \xrightarrow{H^*(\mu_T \otimes 1) \circ H^*(\Phi)^{-1} \circ H^*(c)} H^*(LM) \otimes H^*(LM) \xrightarrow{s \otimes s} H_{S^1}^*(LM) \otimes H_{S^1}^*(LM).$$

**Example.** Let  $M$  be the product of two spheres of odd dimension  $N$ . Models for  $M$ ,  $LM$  and  $LM \times_{S^1} ES^1$  are then given by

$$\begin{aligned} M &: (\wedge(x, y), 0), \\ LM &: (\wedge(x, y, \bar{x}, \bar{y}), 0), \\ LM \times_{S^1} ES^1 &: (\wedge(x, y, \bar{x}, \bar{y}, u), D), Du = D(\bar{x}) = D(\bar{y}) = 0, Dy = u\bar{y}, Dx = u\bar{x}. \end{aligned}$$

A set of cocycles representing a basis of the vector space  $\tilde{H}_{S^1}^*(LM) := H_{S^1}^*(LM)/\mathbb{Q}u$  is composed of the elements ([2])

$$\begin{cases} e_{a,b} = \bar{x}^a \bar{y}^b, & (a, b) \in \mathbb{N}^2 - (0, 0) \\ f_{a,b} = (y\bar{x} - x\bar{y})\bar{x}^a \bar{y}^b, & (a, b) \in \mathbb{N}^2. \end{cases}$$

The vector space  $\tilde{H}^*(LM)$  has the following basis

$$\begin{pmatrix} e_{a,b} \\ f_{a,b} \end{pmatrix}, \quad \begin{cases} e'_{a,b} = xy\bar{x}^a \bar{y}^b, & (a, b) \in \mathbb{N}^2, \\ f'_{a,b} = x\bar{x}^a \bar{y}^b, & (a, b) \in \mathbb{N}^2 \\ f''_b = y\bar{y}^b, & b \in \mathbb{N} \end{cases}.$$

From the above description of the Gysin sequence we deduce  $H^*(p)(e_{a,b}) = e_{a,b}$ ,  $H^*(p)(f_{a,b}) = f_{a,b}$ ,  $M'(f'_{a,b}) = e_{a+1,b}$ ,  $M'(e'_{a,b}) = f_{a,b}$ ,  $M'(f''_b) = e_{0,b+1}$ ,  $M'(e_{a,b}) = 0$ , and  $M'(f_{a,b}) = 0$ .

To fix signs, denote by  $xy$  the fundamental class of  $M$ . A straightforward computation shows that

$$\begin{aligned} b^\vee(u^r) &= 0 \\ b^\vee(\bar{x}^p \bar{y}^q) &= \sum_{r=0}^p \sum_{s=0}^q \binom{p}{r} \binom{q}{s} (\bar{x}^r \bar{y}^{s+1} \otimes \bar{x}^{p-r+1} \bar{y}^{q-s} - \bar{x}^{r+1} \bar{y}^s \otimes \bar{x}^{p-r} \bar{y}^{q-s+1}) \\ b^\vee(f_{p,q}) &= (f_{0,0} \otimes 1 + 1 \otimes f_{0,0})(\bar{x} \otimes \bar{y} - \bar{y} \otimes \bar{x}) \cdot b^\vee(\bar{x}^p \bar{y}^q) \end{aligned}$$

To describe the string bracket in  $\mathcal{H}_*(M)$  we choose dual basis  $t_r$ ,  $a_{p,q}$  and  $b_{p,q}$  of  $u^r$ ,  $\bar{x}^p \bar{y}^q$  and  $f_{p,q}$ . In that basis the string bracket satisfies

$$\begin{aligned} [b_{k,t}, a_{l,m}] &= \binom{k+l}{k} \binom{m+t}{t} \frac{km-lt}{(k+l)(t+m)} b_{k+l-1, t+m-1}, \\ [a_{k,t}, a_{l,m}] &= \binom{k+l}{k} \binom{m+t}{t} \frac{lt-km}{(k+l)(t+m)} a_{k+l-1, t+m-1}, \\ [b_{r,s}, b_{m,n}] &= 0 \end{aligned}$$

In particular the string Lie algebra  $\mathcal{H}_*(M)$  is not nilpotent, since for instance  $[a_{1,1}, a_{r,s}] = (r-s)a_{r,s}$ .

## 5 Hochschild cohomology of a Lie algebra

Let  $(L, d_L)$  be a differential graded Lie algebra. The chain map

$$\varphi : C_*L \rightarrow \mathcal{B}(UL)$$

defined by

$$\varphi(sx_1 \wedge \cdots \wedge sx_n) = \sum_{\sigma \in \Sigma_n} \varepsilon_\sigma [x_{\sigma(1)} | \cdots | x_{\sigma(n)}],$$

is a quasi-isomorphism of coalgebras ([8], Proposition 22.7), that can be extended for each right  $L$ -module  $P$  and each left  $L$ -module  $N$  into a quasi-isomorphism of complexes

$$\Phi = 1 \otimes \varphi \otimes 1 : C_*(P; L; N) \rightarrow \mathcal{B}(P, UL, N).$$

In particular when  $P = N = UL$ ,  $\Phi$  is a quasi-isomorphism of  $UL$ -bimodules.

By composition with  $\Phi$  we get a quasi-isomorphism of complexes

$$\text{Hom}(\Phi, UL) : \text{Hom}_{(UL)^e}(\mathcal{B}(UL; UL; UL), UL) \rightarrow \text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL).$$

The linear isomorphism  $\text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) = \text{Hom}_{\mathbb{Q}}(C_*L, UL)$  gives to the right hand side complex an algebra structure:

$$f \cdot g : C_*L \xrightarrow{\Delta} C_*L \otimes C_*L \xrightarrow{f \otimes g} UL \otimes UL \xrightarrow{\mu} UL.$$

The left hand side complex is equipped with the Gerstenhaber usual product.

**Theorem 4.** *The correspondence  $\text{Hom}(\Phi, UL)$  is a quasi-isomorphism of differential graded algebras inducing in homology an isomorphism of graded algebras*

$$HH^*(UL, UL) \cong H^* \left( \text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) \right).$$

**Proof.** We form the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{(UL)^e}(\mathcal{B}(UL; UL; UL), UL) & \xrightarrow{\text{Hom}(\Phi, UL)} & \text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) \\ \downarrow \cong & & \downarrow \cong \\ \text{Hom}(\mathcal{B}(UL), UL) & \xrightarrow{\text{Hom}(\varphi, UL)} & \text{Hom}(C_*(L), UL). \end{array}$$

Since  $\varphi$  is a morphism of coalgebras,  $\text{Hom}(\varphi, UL)$  is a morphism of algebras.  $\square$

The canonical isomorphism

$$\text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) \cong \text{Hom}_{\mathbb{Q}}(C_*(UL; L; UL) \otimes_{(UL)^e} (UL)^\vee, \mathbb{Q})$$

gives then clearly the isomorphism

$$HH^*(UL; UL) \cong \text{Hom} \left( H_* \left( C_*(UL; L; UL) \otimes_{(UL)^e} (UL)^\vee \right), \mathbb{Q} \right).$$

We equip  $UL$  and  $(UL)^\vee$  with the left adjoint representation,

$$l \cdot x = [l, x], (l \cdot f)(x) = -(-1)^{|l| \cdot |f|} f([l, x]), \quad l \in L, x \in UL, f \in (UL)^\vee,$$

and, to avoid confusion, we denote these modules respectively by  $(UL)_a$  and  $(UL)_a^\vee$ . Then a straightforward computation shows

**Lemma 1.** *The linear isomorphism*

$$C_*(L, (UL)_a^\vee) = C_*(UL; L; UL) \otimes_{(UL)^e} (UL)^\vee$$



is an isomorphism of chain complexes.

The chain complex  $C_*(L; (UL)_a^\vee)$  is a coalgebra that is the tensor product of the coalgebras  $C_*(L)$  and  $(UL)^\vee$ . The dual algebra is the cochain algebra  $C^*(L; (UL)_a^\vee)$ . Since  $\text{Hom}(C_*(L; (UL)_a^\vee), \mathbb{Q}) = \text{Hom}_{UL}(C_*(L; UL), (UL)_a) = C^*(L; (UL)_a)$ , we deduce

**Corollary 1.** *Let  $L$  be a differential graded Lie algebra, then we have isomorphisms of algebras*

$$HH^*(UL, UL) \cong \text{Hom}(H_*(C_*(L; (UL)_a^\vee)), \mathbb{Q}) \cong H^*(C^*(L; (UL)_a)).$$

## 6 Lie models and the cap product $C^*(L; M) \rightarrow C_{m-*}(L; M)$

### 6.1 Minimal Lie models

Let  $S$  be a simply connected space. By Quillen theory ([17],[14],[8]), we can associate to  $S$  a differential graded Lie algebra  $L = \mathcal{L}_S$ , that is free as a graded Lie algebra,  $L = \mathbb{L}(V)$ , with a differential  $d$  satisfying  $d(V) \subset \mathbb{L}^{\geq 2}(V)$  (the sub vector space generated by the brackets of length at least two).

The differential graded Lie algebra  $L$  contains all the rational homotopy type of  $S$ . In particular there exists a quasi-isomorphism from the enveloping algebra on  $L$  into the chain algebra  $C_*(\Omega S; \mathbb{Q})$ ,

$$\varphi_S : UL \xrightarrow{\cong} C_*(\Omega S; \mathbb{Q}).$$

Moreover a differential graded Lie algebra  $L$  is a Lie model for  $S$  if and only if the cochain algebra on  $L$ ,  $C^*L$ , is a Sullivan model of  $S$ .

### 6.2 A Lie model for $LM$

Starting from a differential graded Lie algebra  $L$  we construct another differential graded Lie algebra  $L^S$ , that is a Lie model for  $LM$  when  $L$  is a Lie model for  $M$ .

The differential graded Lie algebra  $L^S$  is defined as follows

$$L_n^S = L_n \oplus \overline{L}_n, \quad \overline{L}_n = L_{n+1}, \quad d\overline{x} = -\overline{dx}, \quad (-1)^{|a|}[a, \overline{b}] = \overline{[a, b]}, \quad [\overline{a}, \overline{b}] = 0.$$

In particular  $\overline{L}$  is an abelian sub Lie algebra. By the Poincaré-Birkhoff-Witt Theorem the chain coalgebra  $C_*\overline{L} = \wedge(s\overline{L})$  is isomorphic to  $UL$  as a differential graded coalgebra:

$$\varphi : C_*(\overline{L}) \xrightarrow{\cong} UL, \quad \varphi(s\overline{x}_1 \wedge \cdots \wedge s\overline{x}_n) = x_1 \cdots x_n.$$

Since  $C_*(L^S) \cong C_*(L) \otimes C_*(\overline{L})$ , we have by direct computation

**Lemma 2.** *The morphism  $1 \otimes \varphi : C_*(L^S) \rightarrow C_*(L; (UL)_a)$  is an isomorphism of coalgebras.*

The relation with the free loop space is the content of Theorem 5.

**Theorem 5.** *If  $L$  is a Lie model for  $M$ , then  $L^S$  is a Lie model for  $LM$ , and a Sullivan model for  $LM$  is given by the commutative differential graded algebra  $C^*(L; (UL)_a^\vee)$ .*

**Proof.** Since  $L$  is a Lie model for  $M$ ,  $C^*(L) = (\wedge V, d)$ , with  $V = s(L^\vee)$  ([8], p. 320), is a Sullivan model for  $M$ . We remark that

$$C^*(L^S) = (\wedge(V \oplus sV), D)$$

with  $D(v) = d(v)$  and  $D(sv) = -sd(v)$ . Therefore  $L^S$  is a Lie model for  $LM$ .

The last assertion follows from the following sequence of isomorphisms of differential graded algebras

$$\begin{aligned} C^*(L^S) &\cong \text{Hom}(C_*(L; (UL)_a), \mathbb{Q}) = \text{Hom}(C_*(L; UL) \otimes_{UL} (UL)_a, \mathbb{Q}) \\ &= \text{Hom}_{UL}(C_*(L; UL), (UL)_a^\vee) = C^*(L; (UL)_a^\vee). \end{aligned}$$

□

### 6.3 A cochain model for the path composition $LM \times_M LM \rightarrow LM$

The multiplication  $\mu : UL \otimes UL \rightarrow UL$  is a morphism of  $UL$ -modules for the adjoint representation. Here  $UL$  acts diagonally by adjunction on  $UL \otimes UL$ . Therefore we have a morphism of cochain complexes

$$C^*(L; \mu^\vee) : C^*(L; (UL)_a^\vee) \rightarrow C^*(L; (UL)_a^\vee \otimes (UL)_a^\vee).$$

**Theorem 6.** *When  $L$  is a Lie model for  $M$ , then the morphism  $C^*(L; \mu^\vee)$  is a morphism of commutative differential graded algebras and is a Sullivan model for the path composition  $LM \times_M LM \rightarrow LM$ .*

**Proof.** The complex  $C_*(\mathbb{Q}; L; UL)$  is a differential graded coalgebra whose coproduct is the tensor product of the coproduct on  $C_*L$  and the coproduct on  $UL$ . The diagonal  $\Delta$  on  $C_*L$  induces a quasi-isomorphism of differential graded coalgebras

$$\Delta' : C_*L \rightarrow C_*(\mathbb{Q}; L; UL) \otimes_{UL} C_*(UL; L; \mathbb{Q})$$

defined by  $\Delta'(x) = \sum_i x_i \otimes 1 \otimes x'_i$  when  $\Delta(x) = \sum_i x_i \otimes x'_i$ .

The dual map  $(\Delta')^\vee : C^*L \otimes (UL)^\vee \otimes C^*L \rightarrow C^*L$  is a quasi-isomorphism of differential graded algebras, and the injection  $1 \otimes \varepsilon' \otimes 1 : C^*L \otimes C^*L \rightarrow C^*L \otimes (UL)^\vee \otimes C^*L$  is a relative Sullivan model for the product  $\mu : C^*L \otimes C^*L \rightarrow C^*L$ . Here  $\varepsilon : UL \rightarrow \mathbb{Q}$  denotes the augmentation of  $UL$  and  $\varepsilon'$  the dual map.

A cochain model for the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\Delta} & M \times M & \xleftarrow{\Delta} & M \\ \parallel & & \uparrow \rho & & \parallel \\ M & \xrightarrow{\Delta} & M \times M \times M & \xleftarrow{\Delta} & M \end{array}$$

is given by

$$\begin{array}{ccccc} C^*L & \xleftarrow{\mu} & C^*L \otimes C^*L & \xrightarrow{\mu} & C^*L \\ \parallel & & 1 \otimes \tau \otimes 1 \downarrow & & \parallel \\ C^*L & \xleftarrow{(\mu \otimes 1) \circ \mu} & C^*L \otimes C^*L \otimes C^*L & \xrightarrow{(\mu \otimes 1) \circ \mu} & C^*L. \end{array}$$

Here  $\rho(a, b, c) = (a, c)$  and  $\tau$  denotes the unit  $\mathbb{Q} \rightarrow C^*L$ . In order to replace the maps  $\mu$  and  $(\mu \otimes 1) \circ \mu$  by relative models, we decompose the right hand square into two squares

$$\begin{array}{ccccccc} C^*L \otimes C^*L & \xrightarrow{1 \otimes \varepsilon' \otimes 1} & C^*L \otimes (UL)^\vee \otimes C^*L & \xrightarrow{(\Delta')^\vee} & C^*L \\ 1 \otimes \tau \otimes 1 \downarrow & & \downarrow 1 \otimes \nabla' \otimes 1 & & \parallel \\ C^*L \otimes C^*L \otimes C^*L & \xrightarrow{\bar{\varepsilon}} & C^*L \otimes (UL)^\vee \otimes C^*L \otimes (UL)^\vee \otimes C^*L & \xrightarrow{((\Delta')^\vee \otimes 1)(\Delta')^\vee} & C^*L \end{array}$$

where  $C^*L \otimes (UL)^\vee \otimes C^*L = (C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee$ ,  $\bar{\varepsilon} = 1 \otimes \varepsilon' \otimes 1 \otimes \varepsilon' \otimes 1$ ,  $C^*L \otimes (UL)^\vee \otimes C^*L \otimes (UL)^\vee \otimes C^*L = (C_*(L; UL) \otimes_{UL} C_*(UL; L; UL) \otimes_{UL} C_*(UL; L))^\vee$ ,  $\nabla$  is the coproduct on  $(UL)^\vee$  and  $\nabla'$  is the composition

$$\nabla' : (UL)^\vee \xrightarrow{\nabla} (UL)^\vee \otimes \mathbb{Q} \otimes (UL)^\vee \xrightarrow{1 \otimes \tau \otimes 1} (UL)^\vee \otimes C^*L \otimes (UL)^\vee = (C_*(UL; L; UL))^\vee.$$

Now we consider the following commutative diagram in which the front face and the back face are fiber squares. By section 3.5, this implies that  $C^*(L; \mu^\vee)$  is a model for the path composition  $LM \times_M LM \rightarrow LM$ .

$$\begin{array}{ccccc}
C^*(L; (UL)_a) & \xleftarrow{\quad} & (C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee & & \\
\uparrow & \searrow C^*(L; \mu^\vee) & \uparrow & \searrow 1 \otimes \nabla' \otimes 1 & \\
& C^*(L; (UL)_a \otimes (UL)_a) & \xleftarrow{1 \otimes \varepsilon' \otimes 1} & E & \\
\uparrow & \mu & \uparrow & \uparrow \bar{\varepsilon} & \\
C^*L & \xleftarrow{id_{C^*L}} & C^*L \otimes C^*L & & \\
\downarrow & \downarrow & \downarrow & \downarrow 1 \otimes \tau \otimes 1 & \\
& C^*L & \xleftarrow{(\mu \otimes 1)\mu} & C^*L \otimes C^*L \otimes C^*L & 
\end{array}$$

with  $E = (C_*(L; UL) \otimes_{UL} C_*(UL; L; UL) \otimes_{UL} C_*(UL; L))^\vee$ .  $\square$

Denote by  $L_1^S$  and  $L_2^S$  two copies of  $L^S$ , and  $L^T = L_1^S \oplus_L L_2^S = L \oplus \bar{L}_1 \oplus \bar{L}_2$ . We denote by  $\pi : L^T \rightarrow L^S$  the projection obtained by mapping identically each  $L_i^S$  to  $L^S$ . We observe that  $C_*(L^T)$  is isomorphic to  $C_*(L; (UL)_a^\vee \otimes (UL)_a^\vee)$  and that the following diagram commutes

$$\begin{array}{ccccc}
C^*(L; (UL)_a^\vee) & \xrightarrow{C^*(L; \mu^\vee)} & C^*(L; (UL)_a^\vee \otimes (UL)_a^\vee) & & \\
\parallel & & \parallel & & \\
\text{Hom}(C_*(L; (UL)_a), \mathbb{Q}) & \xrightarrow{\text{Hom}(C_*(L; \mu), \mathbb{Q})} & \text{Hom}(C_*(L; (UL)_a \otimes (UL)_a), \mathbb{Q}) & & \\
\parallel & & \parallel & & \\
C^*(L^S) & \xrightarrow{C^*(\pi)} & C^*(L^T) & & 
\end{array}$$

This shows that  $\pi : L^T \rightarrow L^S$  is a Lie model for the path composition  $LM \times_M LM \rightarrow LM$ .

Recall that a coformal space is a space that admits a Sullivan minimal model with a purely quadratic differential. The minimal Sullivan model of a coformal space is the cochain algebra on the Lie algebra  $\pi_*(\Omega M) \otimes \mathbb{Q}$ . Therefore,

**Corollary 2.** *Let  $M$  be a coformal manifold with minimal model  $(\wedge V, d)$ , then a model for the path composition  $LM \times_M LM \rightarrow LM$  is given by*

$$id \otimes \Delta : (\wedge V \otimes \wedge \bar{V}, D) \rightarrow (\wedge V \otimes \wedge \bar{V} \otimes \wedge \bar{V}', D),$$

where  $\Delta$  is the coproduct in  $\wedge \bar{V} = H^*(\Omega M)$  and  $(\wedge V \otimes \wedge \bar{V}, D)$  is the model of the free loop space.

**Example.** Let  $M$  be the 11-dimensional manifold obtained by taking the pullback of the tangent sphere bundle to  $S^6$  along the map  $f : S^3 \times S^3 \rightarrow S^6$  that collapses the 3-skeleton

into a point.

$$\begin{array}{ccc} M & \rightarrow & \tau_S S^6 \\ \downarrow & & \downarrow \\ S^3 \times S^3 & \xrightarrow{f} & S^6 \end{array}$$

The minimal model of  $M$  is

$$(\wedge(x, y, z), d), \quad dx = dy = 0, dt = xz, \quad |x| = |y| = 3, |z| = 5.$$

Thus  $M$  is a coformal space. A model for the path composition  $M^{[0,1]} \times_M M^{[0,1]} \rightarrow M^{[0,1]}$  is given by

$$\varphi : (\wedge(x, y, z, x', y', z', \bar{x}, \bar{y}, \bar{z}), D) \rightarrow (\wedge(x, y, z, x', y', z', x'', y'', z'', \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), D),$$

with  $D(\bar{x}) = x - x'$ ,  $D(\bar{y}) = y - y'$ ,  $D(\bar{z}) = z - z' - \frac{1}{2}\bar{x}(y + y') + \frac{1}{2}(x + x')\bar{y}$ ,  $D(\bar{x}') = x' - x''$ ,  $D(\bar{y}') = y' - y''$ ,  $D(\bar{z}') = z' - z'' - \frac{1}{2}\bar{x}'(y' + y'') + \frac{1}{2}(x' + x'')\bar{y}'$ ,  $\varphi(\bar{x}) = \bar{x} + \bar{x}'$ ,  $\varphi(\bar{y}) = \bar{y} + \bar{y}'$ ,  $\varphi(\bar{z}) = \bar{z} + \bar{z}' + \frac{1}{2}\bar{x}\bar{y}' - \frac{1}{2}\bar{x}'\bar{y}$ . The induced model for the path composition  $LM \times_M LM \rightarrow LM$  is then given by

$$\varphi : (\wedge(x, y, z, \bar{x}, \bar{y}, \bar{z}), D) \rightarrow (\wedge(x, y, z, \bar{x}, \bar{y}, \bar{z}, \bar{x}', \bar{y}', \bar{z}'), D),$$

with  $D(\bar{x}) = 0$ ,  $D(\bar{y}) = 0$ ,  $D(\bar{z}) = -\bar{x}y + x\bar{y}$ ,  $D(\bar{x}') = 0$ ,  $D(\bar{y}') = 0$ ,  $D(\bar{z}') = -\bar{x}'y + x\bar{y}'$ ,  $\varphi(\bar{x}) = \bar{x} + \bar{x}'$ ,  $\varphi(\bar{y}) = \bar{y} + \bar{y}'$ ,  $\varphi(\bar{z}) = \bar{z} + \bar{z}' + \frac{1}{2}\bar{x}\bar{y}' - \frac{1}{2}\bar{x}'\bar{y}$ .

## 6.4 The cap product

Let  $L$  be a differential graded Lie algebra and  $N$  be a left  $UL$ -module. Then  $N$  is a right  $UL$ -module for the action defined by

$$n \cdot x := -(-1)^{|n| \cdot |x|} x \cdot n.$$

This gives the opportunity to consider in the same time a module as a left module and as a right module.

Let  $c = \sum_i s x_{i_1} \wedge \cdots \wedge s x_{i_q}$  be a cycle of degree  $m$  in  $C_q L$ . We define the cap product by  $c$ ,

$$cap_c : C^{q-r}(L; N) = \text{Hom}_{UL}(C_{q-r}(L; UL), N) \rightarrow C_r(L; N)$$

by mapping an element  $f$  to the element  $f \cap c$  defined by

$$f \cap c = (-1)^m \sum_i \sum_{\sigma \in \Sigma_q} (-1)^{|f| \cdot (|s x_{\sigma(i_1)} \wedge \cdots \wedge s x_{\sigma(i_r)}|)} \varepsilon_\sigma s x_{\sigma(i_1)} \wedge \cdots \wedge s x_{\sigma(i_r)} \otimes f(s x_{\sigma(i_{r+1})} \cdots s x_{\sigma(i_q)}).$$

Remark that in  $C^*(L; N)$ ,  $N$  is considered as a right  $UL$ -module and in  $C_*(L; N)$ ,  $N$  is viewed as a left  $UL$ -module. By a standard computation we then have

**Lemma 3.** *The morphism  $cap_c$  is a natural morphism of complexes: If  $g : P \rightarrow N$  be a morphism of left  $UL$ -modules, then the following diagram is commutative*

$$\begin{array}{ccccc} C_*(L; P) & \xrightarrow{C_*(L; g)} & C_*(L; N) \\ cap_c \uparrow & & cap_c \uparrow \\ C^*(L; P) & \xrightarrow{C^*(L; g)} & C^*(L; N) \end{array}$$

In particular since the multiplication  $\mu : UL \otimes UL \rightarrow UL$ , and its dual co-multiplication  $\mu^\vee$  are morphisms of  $UL$ -modules for the adjoint representation, we have

**Theorem 7.** *The following diagram commutes*

$$\begin{array}{ccccc}
C_*(L; (UL)_a^\vee) & \xrightarrow{C_*(L; \mu^\vee)} & C_*(L; (UL)_a^\vee \otimes (UL)_a^\vee) \\
\uparrow \text{cap}_c & & \uparrow \text{cap}_c \\
\text{Hom}_{UL}(C_*(L; UL), (UL)_a^\vee) & \xrightarrow{\text{Hom}(1, \mu^\vee)} & \text{Hom}_{UL}(C_*(L; UL), (UL)_a^\vee \otimes (UL)_a^\vee) \\
\parallel & & \parallel \\
C^*(L; (UL)_a^\vee) & \xrightarrow{C^*(L; \mu^\vee)} & C^*(L; (UL)_a^\vee \otimes (UL)_a^\vee)
\end{array}$$

## 7 Proof of Theorem B

The proof is based on the following diagram

$$\begin{array}{ccccc}
C_*(L; (UL)_a^\vee) & \xrightarrow{C_*(L; \mu^\vee)} & C_*(L; (UL \otimes UL)_a^\vee) & \xrightarrow{C_*(\Delta; id)} & C_*(L; (UL)_a^\vee) \otimes C_*(L; (UL)_a^\vee) \\
\uparrow \text{cap}_c & & \uparrow \text{cap}_c & & \\
C^*(L; (UL)_a^\vee) & \xrightarrow{C^*(L; \mu^\vee)} & C^*(L; (UL \otimes UL)_a^\vee) & & \uparrow (\text{cap}_c) \otimes (\text{cap}_c) \\
& & \uparrow \cong & & \\
& & C^*(L; (UL)_a^\vee) \otimes C^*(L; (UL)_a^\vee) \otimes \wedge \bar{V} & \xrightarrow{\mu_T \otimes 1} & C^*(L; (UL)_a^\vee) \otimes C^*(L; (UL)_a^\vee),
\end{array}$$

where  $c$  is a cycle representing the orientation class in homology. The dual of the upper row induces the Gerstenhaber product in Hochschild cohomology (Section 5, Corollary 1), the left hand square is commutative by Theorem 7, and the bottom line induces in cohomology the dual of the Chas-Sullivan product (Theorem A and Theorem 6). Since  $H^*(L; \mathbb{Q})$  satisfies Poincaré duality, a classical spectral sequence argument shows that  $\text{cap}_c$  is a quasi-isomorphism. We will prove in this chapter (Theorem 8) that the right hand square commutes in homology. This will end the proof of Theorem A.  $\square$

### 7.1 The cap product and the diagonal class

Let  $L$  be a Lie model for  $M$ . The algebra  $H^*(C^*(L)) = H^*(M; \mathbb{Q})$  is a Poincaré duality algebra with a fundamental class in degree  $m$ . According to the notations of section 3.4, we denote by  $u = u_M$  a cycle in  $C_*(L)$  whose class is a generator of  $H_m(C_*L)$  and by  $\alpha_i$ ,  $i = 1, \dots, p$ , a graded basis of  $H^*(L; \mathbb{Q})$ . The Poincaré dual basis  $\alpha_i^\#$  is defined by

$$\langle \alpha_i \cup \alpha_j^\#; [u_M] \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad ([16], \text{Theorem 11.10})$$

We denote by  $a_i$  and  $a_i^\#$  cocycles representing the classes  $\alpha_i$  and  $\alpha_i^\#$ . The diagonal cohomology class  $\delta_M$  is then the class of the cocycle  $T = \sum_i (-1)^{|a_i|} a_i \otimes a_i^\#$  ([16], Theorem 11.11).

The purpose of this section consists of the proof of the following theorem.

**Theorem 8.** *With the above notations, the following diagram commutes in homology*

$$\begin{array}{ccc}
C_*(L; (UL)_a^\vee \otimes (UL)_a^\vee) & \xrightarrow{C_*(\Delta; id)} & C_*(L; (UL)_a^\vee) \otimes C_*(L; (UL)_a^\vee) \\
\uparrow (cap_u) & & \\
C^*(L; (UL)_a^\vee \otimes (UL)_a^\vee) & & \uparrow (cap_u) \otimes (cap_u) \\
\Phi \uparrow \cong & & \\
C^*(L; (UL)_a^\vee) \otimes C^*(L; (UL)_a^\vee) \otimes \wedge \bar{V} & \xrightarrow{\mu_T \otimes 1} & C^*(L; (UL)_a^\vee) \otimes C^*(L; (UL)_a^\vee)
\end{array}$$

The morphisms  $\mu_T$  and  $\Phi$  are defined in section 3.4.

For the proof we will use the following general Lemma for modules over differential graded algebras.

**Lemma 4.** *Let  $R$  be a differential graded algebra,  $R \otimes V$  be a semifree  $R$ -module, and  $f, g : N \rightarrow P$  morphisms of right  $R$ -modules. If  $H_*(f) = H_*(g)$  then the maps  $f \otimes 1, g \otimes 1 : N \otimes_R (R \otimes V) \rightarrow P \otimes_R (R \otimes V)$  induce also the same map in homology.*

We also recall the relation between the diagonal class and the cap product with the fundamental class (the proof is a direct computation following the previous definitions).

**Lemma 5.** *With the previous notations the following diagram is commutative*

$$\begin{array}{ccc}
H_*(L; \mathbb{Q}) & \xrightarrow{H_*(\Delta)} & H_*(L \oplus L; \mathbb{Q}) \\
\uparrow (cap_{[u]}) & & \uparrow (cap_{[u]}) \otimes (cap_{[u]}) \\
H^*(L; \mathbb{Q}) & \xrightarrow{\mu_T^*} & H^*(L; \mathbb{Q}) \otimes H^*(L; \mathbb{Q})
\end{array}$$

**Proof of Theorem 8.**

We denote  $R = C^*L \otimes C^*L$ . We remark that  $R \otimes V = C^*(L; (UL)_a^\vee) \otimes C^*(L; (UL)_a^\vee)$  is a semifree  $R$ -module. Remark also that diagram in Theorem 8 is the tensor product of a diagram of  $R$ -module by  $R \otimes V$  over  $R$ . By Lemma 5, this diagram commutes in homology when  $V = \mathbb{Q}$ . It commutes then in homology by Lemma 4, and this achieves the proof of the Theorem.  $\square$

## 7.2 Proof of Lemma 4

Let  $R$  be a differential graded algebra. Each differential  $R$ -module  $N$  admits a semifree resolution, i.e., there exists a semifree  $R$ -module  $S$  and a quasi-isomorphism  $\psi : S \rightarrow N$ . For recall  $S$  is a semifree resolution means that  $S = \sum_{n \geq 0} R \otimes W(n)$  with  $d(W(n)) \subset \sum_{i < n} R \otimes W(i)$ . The sub-complexes  $F^p = \sum_{q \leq p} R \otimes W(q)$  form then a filtration that induces a spectral sequence converging to the cohomology of  $N$ , with  $E_{q,*}^0 = R \otimes W(q)$ .

It is easy to see that each module  $N$  admits a semifree resolution  $P$  such that the associated spectral sequence satisfies  $E_{0,*}^2 = H_*(M)$  as an  $H_*(R)$ -module and  $E_{q,*}^2 = 0$  for  $q > 0$ .

We can assume that  $R \otimes V$  is a semifree resolution satisfying this property. The spectral sequences obtained by filtering  $N \otimes_R R \otimes V$  and  $P \otimes_R R \otimes V$  by the filtration degree in  $V$ ,  $F^p$ , degenerates at the  $E_2$ -term. Since  $E_1(f) = H(f) \otimes id_V$  and  $E_1(g) = H(g) \otimes id_V$ , the maps  $f$  and  $g$  induce the same map at the  $E_1$ -level. Since the spectral sequences degenerate at the  $E_2$ -terms and the morphisms  $f$  and  $g$  are the identity on  $R \otimes V$ ,  $H(f) = H(g)$ .  $\square$

## 8 The intersection morphism

For each subspace  $Z \hookrightarrow M$ , we denote by  $L_Z M$  the space of loops in  $LM$  that originates in  $Z$ . We have thus the commutative pullback diagram

$$\begin{array}{ccc} L_Z M & \xrightarrow{i} & LM \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & M. \end{array}$$

Denote now by  $D$  a closed disk around the base point  $x_0$ . Since  $D$  is contractible,  $L_D M$  is homotopy equivalent to  $D \times \Omega M$ . The restriction morphism  $I : H_*(LM) \rightarrow H_{*-m}(\Omega M)$  is the composition

$$\tilde{H}_*(LM) \rightarrow H_*(LM, L_{M-x_0}M) \xrightarrow{\text{Excision}} H_*(L_D M, L_{S^{m-1}}M) \cong H_m(D, S^{m-1}) \otimes H_{*-m}(\Omega M).$$

Let  $(\wedge V \otimes \wedge \bar{V}, D)$  be a Sullivan model for the free loop space, and let  $\omega \in (\wedge V)^m$  be a cocycle representing the fundamental class. Then the direct sum  $J = (\wedge V)^{>m} \oplus S$ , where  $(\wedge V)^m = \mathbb{Q} \cdot \omega \oplus S$ , is an acyclic differential ideal in  $\wedge V$ ; the quotient map  $q : (\wedge V, d) \rightarrow (A, d) = (\wedge V/J, d)$  is a quasi-isomorphism. We form the tensor product

$$(A \otimes \wedge \bar{V}, D) := (A, d) \otimes_{\wedge V} (\wedge V \otimes \wedge \bar{V}, D).$$

**Lemma 6.** *The injection  $i : \wedge \bar{V} \rightarrow A \otimes \wedge \bar{V}$  defined by  $i(a) = (-1)^m \omega \otimes a$  is a morphism of complexes of degree  $m$  inducing in cohomology the dual of the intersection morphism,  $I^\vee : H^*(\Omega M) \rightarrow H^{*+m}(LM)$ .*

**Proof.** A model for the injection  $L_{M-x_0}M \hookrightarrow LM$  is given by the quotient map  $q : (A \otimes \wedge \bar{V}, D) \rightarrow (A/(\omega) \otimes \wedge \bar{V}, D)$ . Therefore a model for the cochain complexes  $C^*(LM, L_{M-x_0}M)$  is given by  $\text{Ker } q = (\mathbb{Q} \cdot \omega \otimes \wedge \bar{V}, 0)$ . This implies that a model for the composition

$$H^*(\Omega M) \xrightarrow{\cong} H^{*+m}(LM, L_{M-x_0}M) \rightarrow H^{*+m}(LM)$$

is given by the composition

$$\wedge \bar{V} \xrightarrow{a \mapsto (-1)^m \omega \otimes a} \mathbb{Q} \cdot \omega \otimes \wedge \bar{V} \rightarrow H^*(A \otimes \wedge \bar{V}, D).$$

□

**Theorem D.** *There exists an isomorphism of algebras making commutative the diagram*

$$\begin{array}{ccc} H_*(LM) & \xrightarrow{\Phi \cong} & HH^*(C^*M, C^*M) \\ I \downarrow & & \downarrow HH^*(C^*M, \varepsilon) \\ H_*(\Omega M) & \xrightarrow{\Psi} & HH^*(C^*M; \mathbb{Q}) \end{array}$$

where  $\varepsilon : C^*M \rightarrow \mathbb{Q}$  denotes the usual augmentation.

**Proof.**

Denote now by  $L$  the minimal Lie model of  $M$ . The algebra  $C^*(L; (UL)_a^\vee) = C^*(L) \otimes (UL)^\vee$  is then a free model for the free loop space  $LM$ , and we can apply Lemma 6 to this model to have a model for the restriction morphism. Now, since the projection  $q : C^*(L; (UL)_a^\vee) \rightarrow A \otimes (UL)_a^\vee$  is a quasi-isomorphism,  $q$  admits a lifting  $q' : (UL)^\vee \rightarrow C^*(L; (UL)_a^\vee)$ .

Denoting by  $c$  a fundamental class in homology with  $\langle \omega, c \rangle = 1$ , we obtain a commutative diagram

$$\begin{array}{ccc} (UL)^\vee & \xrightarrow{q'} & C^*(L; (UL)_a^\vee) \\ \parallel & & \downarrow (-1)^m - \cap c \\ (UL)^\vee & \xrightarrow{e} & C_*(L; (UL)_a^\vee), \end{array}$$

where  $e(a) = 1 \otimes a$ .

The dual diagram yields in homology to a diagram of algebras whose vertical maps are isomorphisms

$$\begin{array}{ccc} H_*(\Omega M) & \xleftarrow{I} & H_{*+m}(LM) \\ \uparrow & & \uparrow \\ H_*(UL) & \xleftarrow{e'} & H^*(\text{Hom}(C_*(L; (UL)_a^\vee), \mathbb{Q})) = H^*(C^*(L; (UL)_a)) \end{array} \quad (*)$$

Remark that the two  $C_*(L)$ -bimodules  $C_*(L; UL) \otimes_{UL} C_*(UL; L)$  and  $C_*L$  are quasi-isomorphic. Therefore we have by duality a quasi-isomorphism of  $C^*L$ -bimodules  $C^*L \otimes (UL)^\vee \otimes C^*L = (C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee \rightarrow C^*L$ . The Hochschild cochain complexes of  $UL$  and  $C^*L$  are defined by

$$\mathbf{C}^*(UL, UL) = \text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL)$$

$$\mathbf{C}^*(C^*L, C^*L) = \text{Hom}_{(C^*L)^e}((C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee, C^*L).$$

The isomorphisms

$$\text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) \cong \text{Hom}(C_*L, UL)$$

$$\text{Hom}_{(C^*L)^e}((C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee, C^*L) \cong \text{Hom}((UL)^\vee, C^*L)$$

induce differentials on the right hand side terms. Now a simple computation (see for instance [9]) shows that the map  $D$  that associates to a map in  $\text{Hom}(C_*L, UL)$  the dual map in  $\text{Hom}((UL)^\vee, C^*L)$  is an isomorphism of complexes. This induces the following commutative diagram of complexes

$$\begin{array}{ccccc} UL & & \xleftarrow{e'} & & \text{Hom}_{(UL)^e}(C_*(UL; L; UL), UL) \\ \parallel & & & & \parallel \\ UL & & \xleftarrow{e'} & & \text{Hom}(C_*L, UL) \\ \downarrow \cong & & & & \cong \downarrow D \\ \text{Hom}((UL)^\vee, \mathbb{Q}) & & \text{Hom}(\xleftarrow{(UL)^\vee, \varepsilon}) & & \text{Hom}((UL)^\vee, C^*L) \\ \parallel & & & & \parallel \\ \text{Hom}_{(C^*L)^e}((C_*(L; UL) \otimes_{UL} C_*(UL; L))^\vee, \mathbb{Q}) & & \xleftarrow{\mathbf{C}^*(C^*L, \varepsilon)} & & \mathbf{C}^*(C^*L, C^*L) \\ \parallel & & & & \\ \mathbf{C}^*(C^*L, \mathbb{Q}) & & & & \end{array}$$

where  $\varepsilon : C^*L \rightarrow \mathbb{Q}$  denotes the canonical augmentation. The result follows now directly from the induced diagram in cohomology, combined with diagram (\*).  $\square$

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